



# Existence of multiple positive periodic solutions for nonlinear functional difference equations <sup>☆</sup>

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## Abstract

In this paper, we investigate the existence of multiple positive periodic solutions to a class of functional difference equations. We answer the open problems proposed by Y. Raffoul in [Electron. J. Differential Equations 55 (2002) 1–8] and the conditions obtained improve some recent results established there.

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## 1. Introduction

Let  $R$  denote the real numbers, let  $N$  denote the natural numbers, let  $R^+$  denote the nonnegative real numbers. Given  $a < b$  in  $N$ , let  $N[a, b] = \{a, a + 1, \dots, b\}$ . In this paper,

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we apply a cone theoretic fixed point theorem due to Krasnosel'skii [8] to investigate the existence of multiple positive periodic solutions for the functional difference equation

$$x(n+1) = a(n)x(n) + \lambda h(n)f(x(n-\tau(n))), \quad (1.1)$$

where  $a(n)$ ,  $h(n)$  and  $\tau(n)$  are  $T$ -periodic for  $T$  is an integer with  $T \geq 1$ . We assume that  $\lambda$ ,  $a(n)$ ,  $f(x)$  and  $h(n)$  are nonnegative with  $0 < a(n) < 1$  for all  $n \in N[0, T-1]$ .

For the sake of convenience, the hypotheses needed for our criteria are listed as follows:

- ( $H_1$ ) the function  $f: R^+ \rightarrow R^+$  is continuous and there is  $x_n \rightarrow 0$  such that  $f(x_n) > 0$  for  $n = 1, 2, \dots$ ;
- ( $H_2$ ) the function  $h(n) > 0$  for all  $n \in Z$ ;
- ( $H_3$ )  $\sup_{r>0} \min_{r\eta \leq y \leq r} f(y) > 0$ , with  $\eta$  to be defined later;
- ( $H_4$ )  $f \in C([0, \infty), [0, \infty))$  and  $f(y) > 0$  for  $y > 0$ ;
- ( $L_1$ )  $\lim_{y \rightarrow 0} \frac{f(y)}{y} = \infty$ ;
- ( $L_2$ )  $\lim_{y \rightarrow \infty} \frac{f(y)}{y} = \infty$ ;
- ( $L_3$ )  $\lim_{y \rightarrow 0} \frac{f(y)}{y} = 0$ ;
- ( $L_4$ )  $\lim_{y \rightarrow \infty} \frac{f(y)}{y} = 0$ ;
- ( $L_5$ )  $\lim_{y \rightarrow 0} \frac{f(y)}{y} = l$  with  $0 < l < \infty$ ;
- ( $L_6$ )  $\lim_{y \rightarrow \infty} \frac{f(y)}{y} = L$  with  $0 < L < \infty$ .

Equation (1.1) has recently been studied by Y. Raffoul [7]. Some sufficient conditions for the existence of positive periodic solutions of (1.1) are established under some of assumptions above, and the following open problems are posed: Assume that ( $H_1$ ) and ( $H_2$ ) hold, what can be said about Eq. (1.1) when the conditions ( $L_1$ ) and ( $L_2$ ) or ( $L_3$ ) and ( $L_4$ ) are satisfied? This paper aims to solve the above two problems, and we find some results in [7] are the immediate corollaries of the consequences obtained in this paper. The existence of multiple positive periodic solutions of nonlinear functional differential equations has been studied extensively earlier. We cite some appropriate references here [1–3]. In recent years this research area for difference equations has been well developed due to the realization that difference equations are important in application, for example, see [3–6] and references therein. We are particularly motivated by the work of B. Liu [2] on functional differential equations and the work of Y. Raffoul [7] on nonlinear functional difference equations. Throughout this paper, we denote the product of  $y(n)$  from  $n = a$  to  $n = b$  by  $\prod_{n=a}^b y(n)$  with the understanding that  $\prod_{n=a}^b y(n) = 1$  for all  $a > b$ .

## 2. Main results

To prove our main theorems, we first give the following lemmas.

**Lemma 2.1** (Krasnosel'skii [8]). *Let  $B$  be a Banach space, and let  $v$  be a cone in  $B$ . Suppose  $\Omega_1$  and  $\Omega_2$  are open subsets of  $B$  such that  $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$  and suppose that*

$$\Gamma: v \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow v$$

is a completely continuous operator such that

- (i)  $\|\Gamma u\| \leq \|u\|$ ,  $u \in v \cap \partial\Omega_1$ , and  $\|\Gamma u\| \geq \|u\|$ ,  $u \in v \cap \partial\Omega_2$ ; or
- (ii)  $\|\Gamma u\| \geq \|u\|$ ,  $u \in v \cap \partial\Omega_1$ , and  $\|\Gamma u\| \leq \|u\|$ ,  $u \in v \cap \partial\Omega_2$ .

Then  $\Gamma$  has a fixed point in  $v \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

Let  $\chi$  be the set of all real  $T$ -periodic sequences. This set endowed with the maximum norm  $\|x\| = \max_{n \in N[0, T-1]} |x(n)|$ , then  $\chi$  is a Banach space.

**Lemma 2.2** [7].  $x(n) \in \chi$  is a solution of Eq. (1.1) if and only if

$$x(n) = \lambda \sum_{u=n}^{n+T-1} G(n, u) h(u) f(x(u - \tau(u))), \quad (2.1)$$

where

$$G(n, u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}, \quad u \in [n, n+T-1]. \quad (2.2)$$

**Proof.** It is clear that (1.1) is equivalent to

$$\Delta \left( \prod_{s=0}^{u-1} a^{-1}(s) x(u) \right) = \lambda h(u) f(x(u - \tau(u))) \prod_{s=0}^u a^{-1}(s).$$

By summing the above equation from  $u = n$  to  $u = n + T - 1$ , we obtain (2.1). Note that the denominator in  $G(n, u)$  is not zero since  $0 < a(n) < 1$  for all  $n \in N[0, T-1]$ , and the above process is invertible, so (2.1) can derive (1.1) and the proof is completed.  $\square$

We have

$$N \equiv G(n, n) \leq G(n, u) \leq G(n, n+T-1) = G(0, T-1) \equiv M$$

for  $n \leq u \leq n+T-1$  and

$$1 \geq \frac{G(n, u)}{G(n, n+T-1)} \geq \frac{G(n, n)}{G(n, n+T-1)} = \frac{N}{M} > 0$$

for each  $x \in \chi$ , define a cone by

$$P = \{x \in \chi: x(n) \geq 0, n \in N \text{ and } x(n) \geq \eta \|x\|\},$$

where  $\eta = N/M$ , clearly  $\eta \in (0, 1)$ . Define a mapping  $\Psi: \chi \rightarrow \chi$  by

$$(\Psi x)(n) = \lambda \sum_{u=n}^{n+T-1} G(n, u) h(u) f(x(u - \tau(u))),$$

where  $G(n, u)$  is given by (2.2). By the nonnegativity of  $\lambda$ ,  $f$ ,  $a$ ,  $h$  and  $G$ ,  $(\Psi x)(n) \geq 0$  on  $[0, T-1]$ . It is clear that  $(\Psi x)(n+T) = (\Psi x)(n)$  and  $\Psi$  is completely continuous on bounded subset of  $P$ . Also, for any  $x \in P$  we have

$$\begin{aligned}
 (\Psi x)(n) &= \lambda \sum_{u=n}^{n+T-1} G(n, u)h(u)f(x(u - \tau(u))) \\
 &\leq \lambda \sum_{u=0}^{T-1} G(0, T-1)h(u)f(x(u - \tau(u))).
 \end{aligned}$$

Thus,

$$\|\Psi x\| = \max_{n \in [0, T-1]} |\Psi x(n)| \leq \lambda \sum_{u=0}^{T-1} G(0, T-1)h(u)f(x(u - \tau(u))).$$

Therefore,

$$\begin{aligned}
 (\Psi x)(n) &= \lambda \sum_{u=n}^{n+T-1} G(n, u)h(u)f(x(u - \tau(u))) \\
 &\geq \lambda N \sum_{u=0}^{T-1} h(u)f(x(u - \tau(u))) \geq \eta \|\Psi x\|.
 \end{aligned}$$

That is,  $\Psi P$  is contained in  $P$ . In what follows, we let

$$A = \max_{0 \leq u \leq T-1} \sum_{n=0}^{T-1} G(n, u)h(u), \quad B = \min_{0 \leq u \leq T-1} \sum_{n=0}^{T-1} G(n, u)h(u).$$

**Theorem 2.3.** Assume that  $(H_1)$ ,  $(H_2)$ ,  $(L_1)$  and  $(L_2)$  hold. Then, for any  $\lambda \in (0, \lambda^*)$ , where

$$\lambda^* = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0 \leq y \leq r} f(y)},$$

Eq. (1.1) has at least two positive periodic solutions.

**Proof.** Let

$$q(r) = \frac{r}{A \max_{0 \leq y \leq r} f(y)},$$

by  $(H_1)$ , we have that  $q \in C((0, \infty), (0, \infty))$ . By  $(L_1)$  and  $(L_2)$ , we know further that  $\lim_{r \rightarrow 0} q(r) = \lim_{r \rightarrow \infty} q(r) = 0$ . Thus, there exists  $r_0 > 0$  such that  $q(r_0) = \max_{r>0} q(r) = \lambda^*$ . For any  $\lambda \in (0, \lambda^*)$ , by the intermediate value theorem, there exists  $\rho_0 \in (0, r_0)$  such that  $q(\rho_0) = \lambda$ . Thus, we have  $f(y) \leq \frac{\rho_0}{\lambda A}$  for  $y \in [0, \rho_0]$ .

Define

$$\Omega_0 = \{x \in P: \|x\| < \rho_0\}.$$

Then, if  $x \in P \cap \partial\Omega_0$ ,

$$(\Psi x)(n) \leq \lambda \frac{\rho_0}{\lambda A} \sum_{u=0}^{T-1} G(n, u)h(u) \leq \rho_0 = \|x\|.$$

In particular,  $\|\Psi x\| \leq \|x\|$ , for all  $x \in P \cap \partial\Omega_0$ .

By condition  $(L_1)$ , there exists  $0 < \rho_1 < \rho_0$  such that  $f(y) \geq \frac{1}{\lambda\eta B}y$  for  $0 < y \leq \rho_1$ . Define

$$\Omega_1 = \{x \in P: \|x\| < \rho_1\}.$$

Then, if  $x \in P \cap \partial\Omega_1$ ,

$$\begin{aligned} (\Psi x)(n) &\geq \lambda \frac{1}{\lambda\eta B} \sum_{u=n}^{n+T-1} G(n, u)h(u)x(u - \tau(u)) \\ &\geq \frac{1}{B} \|x\| \sum_{u=0}^{T-1} G(n, u)h(u) \geq \|x\|. \end{aligned}$$

In particular,  $\|\Psi x\| \geq \|x\|$ , for all  $x \in P \cap \partial\Omega_1$ .

Again, by condition  $(L_2)$ , there exists  $\rho_2 > \rho_1$  such that  $f(y) \geq \frac{1}{\lambda\eta B}y$  for  $y \geq \rho_2$ .

Define

$$\Omega_2 = \{x \in P: \|x\| < \rho_2\},$$

where  $\rho_2 = \frac{1}{\eta}\rho_0$ .

If  $x \in P \cap \partial\Omega_2$ , then

$$\begin{aligned} (\Psi x)(n) &\geq \lambda \frac{1}{\lambda\eta B} \sum_{u=n}^{n+T-1} G(n, u)h(u)x(u - \tau(u)) \\ &\geq \frac{1}{B} \|x\| \sum_{u=0}^{T-1} G(n, u)h(u) \geq \|x\|. \end{aligned}$$

In particular,  $\|\Psi x\| \geq \|x\|$ , for  $x \in P \cap \partial\Omega_2$ .

By Lemma 2.1, there exist two solutions  $x_1 \in \bar{\Omega}_0/\Omega_1$  and  $x_2 \in \bar{\Omega}_2/\Omega_0$  satisfying  $0 < \|x_1\| < \rho_0 < \|x_2\| < \rho_2$  and this completes the proof of Theorem 2.3.  $\square$

From the arguments in the above proof we have the following consequence.

**Corollary 2.4.** *If  $(H_1)$  and  $(H_2)$  hold, and if either  $(L_1)$  or  $(L_2)$  holds, then for any  $0 < \lambda < \lambda^*$ , Eq. (1.1) has at least one positive periodic solution.*

**Theorem 2.5.** *Assume conditions  $(H_2)$ ,  $(H_4)$ ,  $(L_3)$  and  $(L_4)$  are satisfied. Then, for  $\lambda > \lambda^{**}$ , where*

$$\lambda^{**} = \frac{1}{B} \inf_{r>0} \frac{r}{\min_{\eta r \leq y \leq r} f(y)},$$

*Eq. (1.1) has at least two positive periodic solutions.*

**Proof.** Let  $p(r) = r/B \min_{\eta r \leq y \leq r} f(y)$ . By  $(H_3)$  and  $(H_4)$ , clearly,  $q \in C((0, \infty), (0, \infty))$ . From  $(L_3)$  and  $(L_4)$ , we have that  $\lim_{r \rightarrow 0} p(r) = \lim_{r \rightarrow \infty} p(r) = \infty$ . Thus, there exists  $r_1 > 0$  such that  $p(r_1) = \min_{r_1 > 0} p(r) = \lambda^{**}$ . For any  $\lambda > \lambda^{**}$ , there exists  $\gamma_0 \in (0, r_1)$  such that  $p(\gamma_0) = \lambda$ . Thus, we have  $f(y) \geq \gamma_0/(\lambda B)$  for  $y \in [\eta\gamma_0, \gamma_0]$ .

Define  $\Omega_0 = \{x \in P: \|x\| < \gamma_0\}$ . If  $x \in P \cap \partial\Omega_0$ , then

$$(\Psi x)(n) \geq \lambda \frac{\gamma_0}{\lambda B} \sum_{n=0}^{T-1} G(n, u)h(u) \geq \gamma_0 = \|x\|.$$

In particular,  $\|\Psi x\| \geq \|x\|$ , for all  $x \in P \cap \partial\Omega_0$ ,

By condition  $(L_3)$ , there exists  $0 < \gamma_1 < \gamma_0$  such that  $f(y) \leq \frac{1}{\lambda A}y$  for  $0 < y \leq \gamma_1$ .

Define

$$\Omega_1 = \{x \in P: \|x\| < \gamma_1\}.$$

Then, if  $x \in P \cap \partial\Omega_1$ ,

$$\begin{aligned} (\Psi x)(n) &\leq \lambda \frac{1}{\lambda A} \sum_{u=n}^{n+T-1} G(n, u)h(u)x(n - \tau(u)) \\ &\leq \frac{1}{A} \|x\| \sum_{u=0}^{T-1} G(n, u)h(u) \leq \|x\|. \end{aligned}$$

Next suppose  $(L_4)$  is satisfied, we consider two cases:  $f$  bounded and  $f$  unbounded. The case where  $f$  is bounded is straightforward. If  $f(y)$  is bounded by  $M \geq 0$ , set  $\gamma_2 = \max\{2\gamma_0, \lambda MA\}$ .

If  $x \in P$  and  $\|x\| = \gamma_2$ , then

$$(\Psi x)(n) \leq \lambda M \sum_{u=0}^{T-1} G(n, u)h(u) \leq \lambda MA \leq \gamma_2 = \|x\|.$$

Now assume  $f$  is unbounded. Apply condition  $(L_4)$  and set  $\gamma > \gamma_0$  such that if  $y \geq \gamma$ , then  $f(y) < \frac{\gamma}{\lambda A}$ .

Set  $\gamma_2 = \frac{1}{\eta}\gamma$ . Define

$$\Omega_2 = \{x \in P: \|x\| < \gamma_2\}.$$

Then, if  $x \in P \cap \partial\Omega_2$ ,

$$\begin{aligned} \Psi x(n) &\leq \lambda \frac{1}{\lambda A} \sum_{u=n}^{n+T-1} G(n, u)h(u)x(n - \tau(u)) \\ &\leq \frac{1}{A} \|x\| \sum_{u=0}^{T-1} G(n, u)h(u) \leq \|x\|. \end{aligned}$$

In particular,  $\|\Psi x\| \leq \|x\|$  for  $x \in P \cap \partial\Omega_2$ .

By Lemma 2.1, Eq. (1.1) has two solutions  $x_1 \in \bar{\Omega}_0/\Omega_1$  and  $x_2 \in \bar{\Omega}_2/\Omega_0$  satisfying  $0 < \|x_1\| < \gamma_0 < \|x_2\|$  and the proof of Theorem 2.5 is completed.  $\square$

From the arguments in the above proof we have the following consequence, too.

**Corollary 2.6.** *If  $(H_1)$ – $(H_3)$  hold, and if either  $(L_3)$  or  $(L_4)$  holds, then for any  $\lambda > \lambda^{**}$ , Eq. (1.1) has at least one positive periodic solution.*

In view of two Corollaries 2.4 and 2.6, Theorem 2.4 in [7] is their immediate consequence.

**Corollary 2.7** [7, Theorem 2.4]. *Assume that  $(H_1)$  and  $(H_2)$  are true. Also, if either  $(L_1)$  and  $(L_4)$  hold, or  $(L_2)$  and  $(L_3)$  hold, then (1.1) has at least one positive periodic solution for any  $\lambda > 0$ .*

**Proof.** Suppose first that  $(L_1)$  and  $(L_4)$  hold. If  $\sup_{0 \leq y < \infty} f(y) = M < \infty$ , then  $\lambda^* \geq (1/A) \sup_{r>0} (r/M) = \infty$ . If  $f(y)$  is unbounded, then there exists a sequence  $\{r_n\}$  such that  $f(r_n) = \max_{0 \leq y \leq r_n} f(y)$  and  $\lim_{n \rightarrow \infty} f(r_n) = \infty$ . By  $(L_4)$ , we have

$$\lambda^* \geq \frac{1}{A} \sup_{r_n > 0} \left( \frac{r_n}{f(r_n)} \right) = \infty.$$

Thus, we have proved  $\lambda^* = \infty$ . So, our assertion follows from Corollary 2.4. If  $(L_2)$  and  $(L_3)$  hold, then we have  $\lim_{y \rightarrow \infty} f(y) = \infty$ . Thus,  $(H_3)$  holds. Let  $\{r_n\}$  satisfy  $\lim_{n \rightarrow \infty} r_n = \infty$  and  $f(\sigma r_n) = \min_{\sigma r_n \leq y \leq r_n} f(y)$ . By  $(L_2)$ , we have

$$\lambda^{**} \leq \frac{1}{B} \inf_{r_n > 0} \left( \frac{r_n}{f(\sigma r_n)} \right),$$

thus,  $\lambda^{**} = 0$ . So our assertion follows from Corollary 2.6.  $\square$

Similarly, we can also discuss the equation

$$x(n+1) = a(n)x(n) - \lambda h(n)f(x(n - \tau(n))), \quad (2.3)$$

where  $\lambda$ ,  $a(n)$ ,  $f(x)$  and  $h(n)$  satisfy the same assumptions for (1.1) except that  $a(n) > 1$  for all  $n \in N[0, T-1]$ . Similar theorems and corollaries can be easily stated and proven regarding Eq. (2.3). One can refer to [7].

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